The following are from Gallian, Chapter 3 (6th edition).

In this solution set, \(|a|\) denotes the order of the element \(a\) (i.e. \(\text{ord}(a)\)).

- **# 1:**
  1. **\(\mathbb{Z}_{12}\):**
     * \(|\mathbb{Z}_{12}| = 12;\)
     * \(0| = 1; \ 1| = 12; \ 2| = 6; \ 3| = 4; \ 4| = 3; \ 5| = 12; \ 6| = 2; \ 7| = 12; \ 8| = 3; \ 9| = 4;\)
     * \(10| = 6; \ 11| = 12.\)
  2. **\(U(10)\):**
     * \(|U(10)| = \phi(10) = \phi(2 \cdot 5) = (1)(4) = 4;\)
     * \(1| = 1; \ 3| = 4; \ 7| = 4; \ 9| = 2.\)
  3. **\(U(12)\):**
     * \(|U(12)| = \phi(12) = \phi(2^2 \cdot 3) = (2)(2) = 4;\)
     * \(1| = 1; \ 5| = 2; \ 7| = 2; \ 11| = 2.\)
  4. **\(U(20)\):**
     * \(|U(20)| = \phi(20) = \phi(2^2 \cdot 5) = (2)(4) = 8;\)
     * \(1| = 1; \ 3| = 4; \ 7| = 4; \ 9| = 2; \ 11| = 2; \ 13| = 4; \ 17| = 4; \ 19| = 2.\)
  5. **\(D_4\):**
     * \(|D_4| = 8;\)
     * \(R_0| = 1; \ R_{60}| = 4; \ R_{180}| = 2; \ R_{270}| = 4; \ H| = 2; \ V| = 2; \ D| = 2; \ D'| = 2.\)

- **Observations:** The order of every element divides the order of the group.

- **# 2:**
  1. In \(\mathbb{Q}\), we have \(\left\{\frac{1}{2}\right\} = \{0, \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \pm \frac{5}{2}, \ldots\}\)
  2. In \(\mathbb{Q}^*\), we have \(\left\{\frac{1}{2}\right\} = \{\ldots \frac{1}{2^n}, \ldots, \frac{1}{2}, 1, 2, 4, 8, ..., 2^n, \ldots\}\)

- **# 3:**
  1. In \(\mathbb{Q}\), \(0| = 1.\) All other elements have infinite order.
  2. In \(\mathbb{Q}^*\), \(1| = 1,\) and \(-1| = 2.\) All other elements have infinite order.

- **# 4:** Prove that in any group, an element and its inverse have the same order.
  1. Seeking a contradiction, suppose that this is not true. That is, suppose that there are elements \(a\) and \(a^{-1}\) such that \(|a| = n\) and \(|a^{-1}| = m\), with \(n\) and \(m\) positive integers such that \(n \neq m.\)
  2. Without loss of generality, suppose also that \(m < n.\) Then \(e = e \cdot e = (a^n) \cdot ((a^{-1})^m) = a^{n-m}.\) The consequence that \(a^{n-m} = e\) implies that \(n\) is not the order of \(a,\) contradicting our assumption.
  3. We conclude that \(n = m.\)

- **# 18:** If \(H\) and \(K\) are subgroups of \(G,\) show that \(H \cap K\) is a subgroup of \(G.\)
  1. We use the Two-Step Subgroup Test.
  2. Suppose that \(a, b \in H \cap K.\) That is, \(a, b \in H\) and \(a, b \in K.\) Since \(H\) and \(K\) are subgroups of \(G,\) we know that \(ab \in H\) and \(ab \in K.\) But this means that \(ab \in H \cap K\) as desired.
2. Suppose that \( a \in H \cap K \). That is, \( a \in H \) and \( a \in K \). Since \( H \) and \( K \) are subgroups of \( G \), we know that \( a^{-1} \in H \) and \( a^{-1} \in K \). But this means that \( a^{-1} \in H \cap K \) as desired.

Thus, \( H \cap K \) is a subgroup.

- # 23:
  - Find the centralizer of each member of \( G \).
    
    * \( C(1) = \{G\} \); \( C(2) = \{1, 2, 5, 6\} \); \( C(3) = \{1, 3, 5, 7\} \); \( C(4) = \{1, 4, 5, 8\} \); \( C(5) = \{G\} \); \( C(6) = \{1, 2, 5, 6\} \); \( C(7) = \{1, 3, 5, 7\} \); \( C(8) = \{1, 4, 5, 8\} \).
  - Find \( Z(G) \).
    * \( Z(G) = \{1, 5\} \).
  - Find the order of each element of \( G \).
    * \(|1| = 1\); \(|2| = 2\); \(|3| = 4\); \(|4| = 2\); \(|5| = 2\); \(|6| = 2\); \(|7| = 4\); \(|8| = 2\).
  - Observations: The order of every element divides the order of the group.

- # 26: If \( H \) is a subgroup of \( G \), prove that \( C(H) \) is a subgroup of \( G \).
  - We use the Two-Step Subgroup Test. Let \( x, y \in C(H) \). Then, for all \( h \) in \( H \),

\[
\begin{align*}
h(xy) &= (hx)y \\
      &= (xh)y \\
      &= x(hy) \\
      &= x(yh) \\
      &= (xy)h,
\end{align*}
\]

so \( xy \in C(H) \). Also,

\[
\begin{align*}
hx^{-1} &= exh^{-1} \\
      &= (x^{-1}x)hx^{-1} \\
      &= x^{-1}(xh)x^{-1} \\
      &= x^{-1}(hx)x^{-1} \\
      &= x^{-1}h(xx^{-1}) \\
      &= x^{-1}he \\
      &= x^{-1}h
\end{align*}
\]

so \( x^{-1} \in C(H) \).

This completes the proof.

- # 27: Must the centralizer of an element of a group be Abelian?
  - No. Consider # 17, where \( C(5) = \{G\} \). The group \( G \) is not Abelian, since \( 8 = 2 \cdot 3 \neq 3 \cdot 2 = 4 \).

- # 28: Must the center of a group be Abelian?
  - Yes. Let \( G \) be a group and let \( x, y \in Z(G) \). By definition, if \( x \) is in \( Z(G) \), then \( xy = gx \) for all \( y \in G \). Since \( y \in Z(G) \subset G \), this means that \( xy = yx \).

- # 29: Let \( G \) be an Abelian group with identity \( e \) and let \( n \) be some fixed integer. Prove that the set of all elements of \( G \) that satisfy the equation \( x^n = e \) is a subgroup of \( G \).
Let \( H \) denote the subset of \( G \) satisfying \( x^n = e \). We use the Two-Step Subgroup Test.

1. Let \( x \) and \( y \) be in \( H \). Thus \( x^n = y^n = e \) and

\[
(xy)^n = xy \cdot xy \cdot \cdots \cdot xy = x^n y^n \quad \text{(since \( G \) is abelian)}
\]

\[
= e \cdot e = e.
\]

Therefore \( xy \in H \) as desired.

2. Let \( x \in H \). Then \( x^n = e \) and thus \( (x^{-1})^n = (x^n)^{-1} = e^{-1} = e \). Therefore \( x^{-1} \in H \), and \( H \) is a subgroup of \( G \).

- An example of a group \( G \) in which the set of all elements of \( G \) that satisfy \( x^2 = e \) does not form a subgroup of \( G \):

  * Consider the dihedral group \( D_3 \) from HW \#2. Then \(| F_T | = 2 \) and \(| F_L | = 2 \), but \(| F_T F_L | = | R_{120} | = 3 \).